L30 8.3 Powers and products of Trigonometric functions (續.三角函數的次數和乘積)
Part II: Other trigonometric functions (Conti.)
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8.5 Partial Fractions (部分分式)

If f is twice differentiable on [a,b] and f(a)=f(b)=0, show that

$\int_a^b (x-a)(x-b)f''(x)dx = -2\int_a^b f(x)dx$.

$$\int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx = \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$\Rightarrow \int \sec^3 x dx = \frac{1}{2}\sec x taax + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \sec^6 x dx = \int \sec^2 x(\tan^2 x+1)^2 dx = \int \sec^2 x(\tan^4 x + 2\tan^2 x + 1)dx$$

$$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + x + C$$

(Ⅲ) $\int \tan^m x \sec^n x dx$ (or $\int \cot^m x \csc^n x dx$)

(i) n is even

$$\int \tan^m x \sec^{2k} x dx = \int \tan^m (\tan^2 x+1)^{k-1} \sec^2 x dx = \int u^m (u^2+1)^{k-1} du$$

(ii) n is odd (m is odd)

Q:m 是什麼時候可用 substitution？A:odd。(substitution 是所有積分技巧最簡單的)

$$\int \tan^{2k+1} x \sec^{2l+1} x dx = \int \sec x \tan x (\sec^2 x+1)^k \sec^{2l} x dx = \int (u^2+1)^k u^{2l} du$$

(ii) n is odd (m is even)

$$\int \tan^{2k} x \sec^n x dx = \int (\sec^2 x - 1)^k \sec^n dx = \sum_N \sec^N dx$$

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eg.

① $\int \tan^5 x \sec^4 x dx = \int \tan^5 x(\tan^2 x+1)\sec^2 x dx = \frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 + C$

② $\int \tan^2 x \sec x dx = \int (\sec^2 x - 1)\sec x dx = \int \sec^2 x \cdot \sec x dx - \int \sec x dx$

$= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$

第一步是想法，我們通常不化簡。

③ $\int \tan^5 x \sec^3 x dx = \int \sec x \tan x(\sec^2 x - 1)^2 \sec x^2 dx$

$= \int \sec x \tan x(\sec^6 x - 2\sec^4 x + \sec^2 x)dx = \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 + C$

Ex:415(28.31.37.42.44) substitution 是這個章節想的部分

§ 8.4 Integrals involving $\sqrt{a^2 \pm x^2}$, $\sqrt{x^2 - a^2}$; trigonometric substitution

Integrals involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, and $\sqrt{x^2 - a^2}$ can often be simplified by making a trigonometric substitution.

$a^2 \rightarrow \sin^2 u + \cos^2 = 1$

$a^2 \rightarrow \tan^2 u + 1 = \sec^2 u$

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Let a>0

(i) For $\sqrt{a^2-x^2}$, set x=asinu. Then dx=acosu du and u=$\sin^{-1}\frac{u}{a}$

∴ u∈$[-\frac{\pi}{2},\frac{\pi}{2}]$. Hence $\sqrt{a^2-x^2}=a|\cos u|=a\cos u$ (∵u∈$[-\frac{\pi}{2},\frac{\pi}{2}]$)

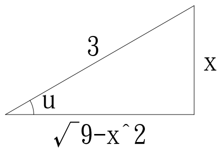
(ii) For $\sqrt{a^2+x^2}$, set x=atanx. Then dx=acosu du and u=$\tan^{-1}\frac{u}{a}$

∴ u∈$(-\frac{\pi}{2},\frac{\pi}{2})$. Hence $\sqrt{a^2-x^2}=a|\sec u|=a\sec u$ (∵u∈$(-\frac{\pi}{2},\frac{\pi}{2})$)

(iii) For $\sqrt{x^2-a^2}$, set x=asecx. Then dx=asec du and u=$\sec^{-1}\frac{u}{a}$

∴ u∈$[0,\frac{\pi}{2})(\frac{\pi}{2},\pi]$. Hence $\sqrt{x^2-a^2}=a|\tan u|$ (∵u∈$[0,\frac{\pi}{2})(\frac{\pi}{2},\pi]$)

eg. ① $\int\frac{dx}{(9-x^2)^{\frac{3}{2}}}=?$



pf: Let x=3sinu, then dx=3cosudu

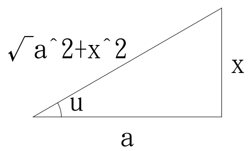
$$\int\frac{dx}{(9-x^2)^{\frac{3}{2}}}=\int\frac{3\cos u du}{(3\cos u)^3}=\frac{1}{9}\int\sec^2 u du=\frac{1}{9}\tan u+C=\frac{1}{9}\frac{x}{\sqrt{9-x^2}}+C$$

eg ② $\int\sqrt{a^2+x^2}dx=?$

pf: Let u=atanu, then dx=asec^{2}udu

$$\int\sqrt{a^2+x^2}dx=\int a\sec u\cdot a\sec^2 u du=a^2\int\sec^3 u du=\frac{a^2}{2}\sec u\tan u+\frac{a^2}{2}\ln|\sec u+\tan u|+C$$

$$=\frac{a^2}{2}\frac{\sqrt{a^2+x^2}}{a}+\frac{a^2}{2}\ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right|+C$$



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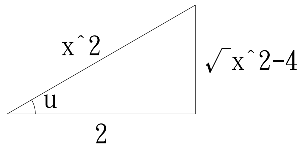
eg ③ $\int_2^4 \frac{dx}{x^2\sqrt{x^2-4}} = ?$



pf: Let x=2secu, then dx=2secutanudu

$$\int_2^4 \frac{dx}{x^2\sqrt{x^2-4}} = \int_0^{\frac{\pi}{3}} \frac{2\sec u \tan u du}{4\sec^2 u|2\tan u|} = \frac{1}{4}\int_0^{\frac{\pi}{3}} \cos u du = \frac{1}{4}\sin u \Big|_0^{\frac{\pi}{3}} = \frac{\sqrt{3}}{8}$$

Ex:P412(13.14.21.26.31.32)

§ Partial Fractions

Thm: $\int \frac{P(x)}{Q(x)} dx$ 一定積得出來，where P and Q are polynomials.

pf:Without loss of genenality(=WlOG)(不失一般性), we may

deg Q>deg P (degree of Q) (i.e. P(x)/Q(x)是真分式)

如果是假分式，先分寫成多項式加真分式。
Q:什麼叫真分式？A:分母最高次數高於分子最高次數。

Step1:將 Q 因式分解成一次式或二次式因式的連乘

Step2:對每一個$(ax+b)^l$給定 $\frac{A_1}{(ax+b)^1} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_l}{(ax+b)^l}$, where $A_1,A,_2A_l \in \mathbb{R}$

對每一個二次式因式$(ax^2+bx+c)^m$給定

$$\frac{A_1x+B_1}{(ax^2+bx+c)^1} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$ $A_1,\cdots,A_m,B_1,\cdots B_m \in \mathbb{R}$

eg. $\frac{3x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$